Flows on Surfaces of Arbitrary Topology

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Alias
Basic Idea

Combine two pieces of code
Flow Model

\[ \frac{\partial u}{\partial t} = -(u \cdot \nabla)u + \nu \nabla^2 u + f \]

\[ \nabla \cdot u = 0 \]

Incompressible Navier-Stokes

Stable Fluids (SIGGRAPH’99)

Demo
Catmull-Clark Surfaces

Catmull and Clark 1978.

Properties:
- Arbitrary Topology
- Smooth (C1) [Reif, Zorin, Peters]
- Exact Evaluation [Siggraph’98]

Demo
Cross-Patch Interactions
Cross-Patch Interactions

Add another layer of cells around each grid
Cross-Patch Interactions

Fill in boundary cells from neighbor patch after each update
Cross-Patch Interactions

for ( i=1 ; i<=N ; i++ ) {
    A[N+1,i] = An[1,i];
}

Easy if Grids aligned
4 Cases / Edge = 16 Cases

[Diagram showing 4 cases with numbers ranging from 0 to 3 in each case]
4 Cases Enough!

Transition code only depends on

\[ k = (4 + e - (f + 2) \mod 4) \mod 4 \]
Simple code

Get neighbor arrays: A0, A1, A2, A3

Compute: k0, k1, k2, k3

for (i = 1; i <= N; i++) {
    A[i, 0] = A0[idx(k0, i, N)];
    A[N+1, i] = A1[idx(k1, 1, i)];
    A[i, N+1] = A2[idx(k2, i, 1)];
    A[0, i] = A3[idx(k3, N, i)];
}
VELOCITY

incorrect

( -1, -0.5 )

correct

( 1, 0.5 )
Are we Done?
Distortions
Distortions (1D)

\[ g = \frac{\partial t}{\partial s} \]

\[ g > 1 \quad \text{and} \quad g < 1 \]
Distortions (1D)

\[ \frac{\partial}{\partial t} = \frac{\partial s}{\partial t} \frac{\partial}{\partial s} = g^{-1} \frac{\partial}{\partial s} \]

\[ \frac{\partial^2}{\partial t^2} = g^{-2} \frac{\partial^2}{\partial s^2} \]
Distortions (2D)

$g_{2,2}$  $g_{1,2}$  $g_{1,1}$
Distortions (2D)

Matrix instead of a scalar:

\[ \mathbf{M} = \begin{pmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{pmatrix} \]

\[ \mathbf{M}^{-1} = \begin{pmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{pmatrix} \]
Distortions (2D)

Operators in “Einstein Notation”:

\[ \nabla^i \equiv g^{i,j} \frac{\partial}{\partial x^j} \]

\[ \nabla^2 = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left( \sqrt{g} g^{i,j} \frac{\partial}{\partial x^j} \right) \]

where \( g = \det (M) \)
Distortions (2D)

\[ g = \det (M) \]
Distortions

No metric

With metric
Distortions

No metric

With metric
Results
Results
Results
Results

Real-Time Demo
Results
Results
Results
Results
Results
Results
Results

+ Reaction-Diffusion
Other Surfaces

Loop Subdivision Surfaces
  Bajaj (2003)

Implicit/Level Set
  Bertalmio + Osher + Shapiro (2001-3)

Meshes
  Desbrun, Alliez, Schroeder, CalTech, etc.
Future Work

Distortions still a problem

Other PDEs, processes, ...

Extensions to 3D...